# *Stochastic models for learning language models (Part 1)*

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*Deep Learning*
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# Outline

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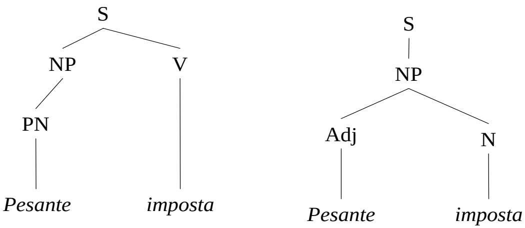
# *Quantitative Models of language structures*

Linguistic structures exhibit syntagmatic information that is crucial for machine learning in Web Mining. The common grammatical modeling framework is the one of (phrase structure) grammars, that can produce often ambiguous readings:

```
1.    S   -> NP   V
2.    S   -> NP
3.    NP  -> PN
4.    NP  -> N
5.    NP  -> Adj N
6.    N   -> "imposta"
7.    V   -> "imposta"
8.   Adj  -> "pesante"
9.    PN  -> "Pesante"
...
```

# *The role of Quantitative Approaches*

“*Pesante imposta*”



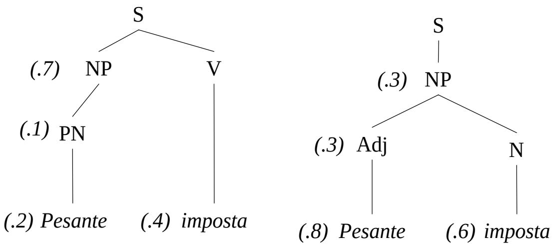
# *The role of Quantitative Approaches*

Weighted grammars are models of (possibly limited) *degrees of grammaticality*. They are meant to deal with a large range of ambiguity problems:

```
1.    S   -> NP   V         .7
2.    S   -> NP             .3
3.    NP  -> PN             .1
4.    NP  -> N              .6
5.    NP  -> Adj N          .3
6.    N   -> imposta        .6
7.    V   -> imposta        .4
8.   Adj  -> Pesante        .8
9.    PN  -> Pesante        .2
```

# *Linguistic Ambiguity and weighted grammars*

"*Pesante imposta*"



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$$\text{prob}(((\text{Pesante})_{PN}\ (\text{imposta})_V)_S) = (.7 \cdot .1 \cdot .2 \cdot .4) = 0.0084$$

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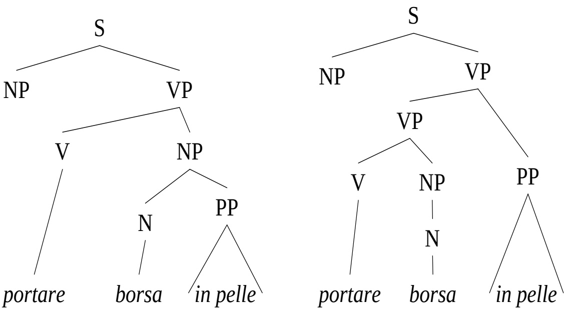
prob(((Pesante)$_{PN}$ (imposta)$_V$)$_S$ )= $(.7 \cdot .1 \cdot .2 \cdot .4) = 0.0084$

prob(((Pesante)$_{Adj}$ (imposta)$_N$)$_S$ )= $(.3 \cdot .3 \cdot .8 \cdot .6) = 0.0432$

# *Syntactic Disambiguation*

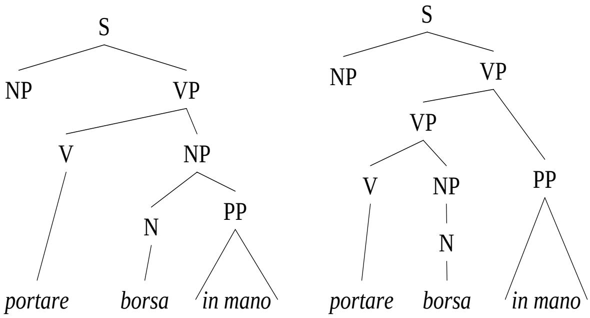
*"portare borsa in pelle"*




*Derivation Trees for a structurally ambiguous sentence*

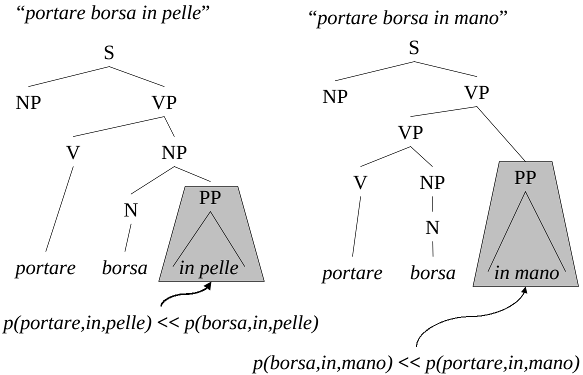
# *Syntactic Disambiguation (cont'd)*

*"portare borsa in mano"*



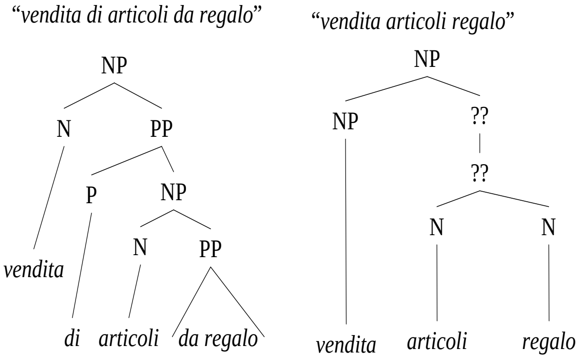

*Derivation Trees for a second structurally ambiguous sentence.*

# *Structural Disambiguation (cont'd)*



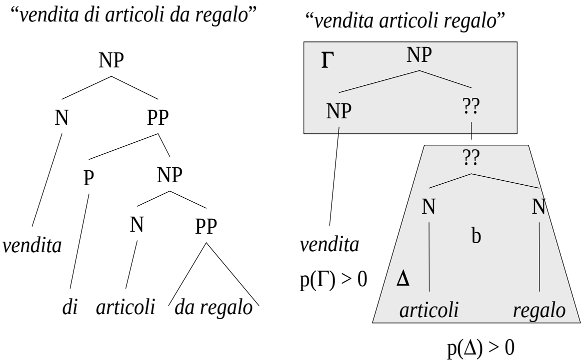

*Disambiguation of structural ambiguity.*

# Tolerance to errors




*An example of ungrammatical but meaningful sentence*

# Error tolerance (cont'd)




*Modeling of ungrammatical phenomena*

# Probability and Language Modeling

- Aims
  - to extend grammatical (i.e. rule-based) models with predictive and disambiguation capabilities
  - to offer theoretically well founded *inductive methods*
  - to develop (not merely) quantitative models of linguistic phenomena

# *Probability and Language Modeling*

- Aims
  - to extend grammatical (i.e. rule-based) models with predictive and disambiguation capabilities
  - to offer theoretically well founded *inductive methods*
  - to develop (not merely) quantitative models of linguistic phenomena
- Methods and Resources:
  - Mathematical theories (e.g. Markov models)
  - Systematic testing/evaluation frameworks
  - Extended repositories of examples of *language in use*
  - Traditional linguistic resources (e.g. "models" like dictionaries)

# Probability and Language Modeling

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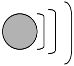

**…, 8, 7, 6, 5, 4, 3, 2, 1**

**…, $W_{i8}$, $W_{i7}$, $W_{i6}$, $W_{i5}$, $W_{i4}$, $W_{i3}$, $W_{i2}$, $W_{i1}$**

# *Probability and Language Modeling*

## *A generative language model*

A random variable $X$ can be introduced so that

# Probability and Language Modeling

### A generative language model

A random variable $X$ can be introduced so that

- It assumes values $w_i$ in the alfabet $A$
- Probability is used to describe the uncertainty on the emitted signal

$$p(X = w_i) \qquad w_i \in A$$

# Probability and Language Modeling

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  - $X$ assumes values in $A$ at each step $i$, i.e. $X_i = w_j$
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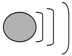
  - $X$ assumes values in $A$ at each step $i$, i.e. $X_i = w_j$
  - probability is $p(X_i = w_j)$
- Constraints: the total probability is for each step:

$$\sum_j p(X_i = w_j) = 1 \qquad \forall i$$



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- Notice that time points can be represented as **states** of the emitting source
- An output $w_i$ can be considered as emitted in a *given state* $X_i$ by the source, and *given a certain* **history**

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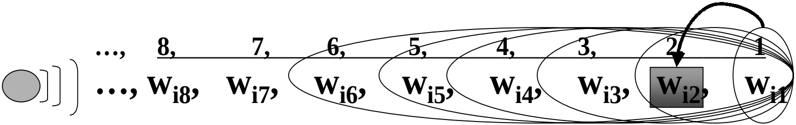

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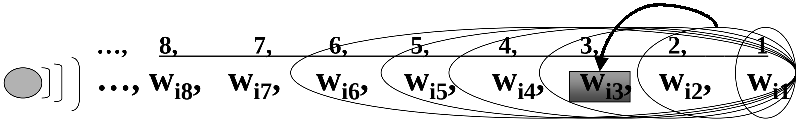
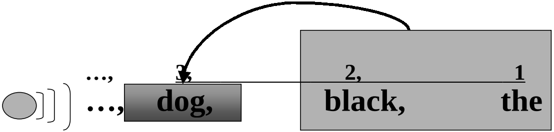
  - $P(X_i = w_i, X_{i-1} = w_{i-1}, \ldots X_1 = w_1) =$
    $= P(X_i = w_i | X_{i-1} = w_{i-1}, X_{i-2} = w_{i-2}, \ldots, X_1 = w_1) \cdot$
    $P(X_{i-1} = w_{i-1}, X_{i-2} = w_{i-2}, \ldots, X_1 = w_1)$



# *Probability and Language Modeling*

## *What's in a state*

$n-1$ preceding words $\Rightarrow$ $n$**-gram language models**

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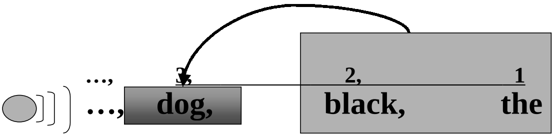


$$p(the, black, dog) = p(dog|the, black)p(black|the)p(the)$$

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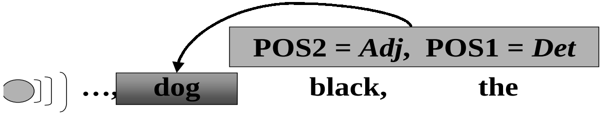
preceding POS tags ⇒ **stochastic taggers**



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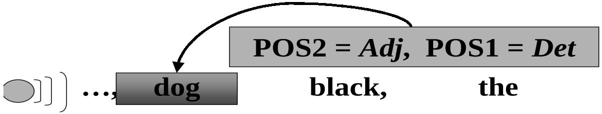


$$p(the_{DT}, black_{ADJ}, dog_N) = p(dog_N | the_{DT}, black_{ADJ}) \ldots$$

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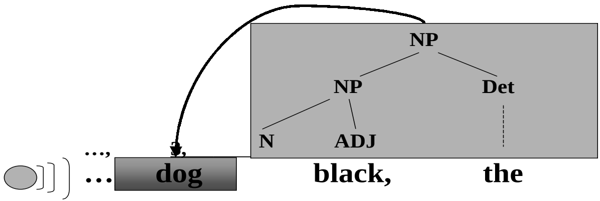
preceding *parses* $\Rightarrow$ **stochastic grammars**



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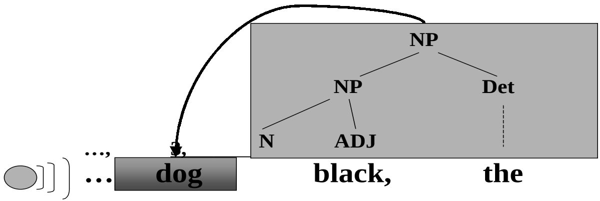


$$p((the_{Det}, (black_{ADJ}, dog_N)_{NP})_{NP}) =$$

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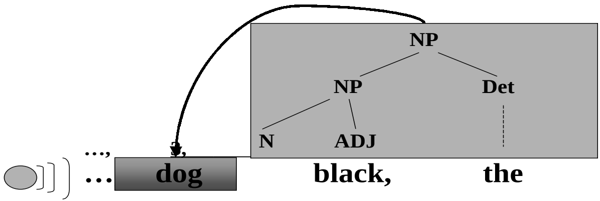


---

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# *Probability and Language Modeling (2)*

- Expressivity
  - The predictivity of a statistical grammar can provide a very good explanatory model of the source language (string)
  - Acquiring information from data has a clear definition, with simple and sound induction algorithms
  - Simple but richer descriptions (e.g. grammatical preferences)
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- Integrating Linguistic Description
  - Start with poor assumptions and approximate as much as possible *what is known* (early evaluate only performance)
  - *Bias* the statistical model since the beginning and check the results on a *linguistic ground*

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- Tools for further analysis of Linguistic Data

# Markov Models

## Markov Models

Suppose $X_1, X_2, \ldots, X_T$ form a sequence of random variables taking values in a countable set $W = p_1, p_2, \ldots, p_N$ (State space).

- Limited Horizon Property:
  $P(X_{t+1} = p_k | X_1, \ldots, X_t) = P(X_{t+1} = k | X_t)$

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- Time invariant:
  $P(X_{t+1} = p_k | X_t = p_l) = P(X_2 = p_k | X_1 = p_l) \qquad \forall t (> 1)$

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It follows that the sequence of $X_1, X_2, \ldots, X_T$ is a **Markov chain**.

# Representation of a Markov Chain

## Markov Models: Matrix Representation

- A (transition) matrix A:

$$a_{ij} = P(X_{t+1} = p_j | X_t = p_i)$$

Note that $\forall i,j \quad a_{ij} \geq 0$ and $\forall i \quad \sum_j a_{ij} = 1$

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- Initial State description (i.e. probabilities of initial states):

$$\pi_i = P(X_1 = p_i)$$

Note that $\sum_{j=1}^{n} \pi_j = 1$.

# Representation of a Markov Chain

Graphical Representation (i.e. Automata)

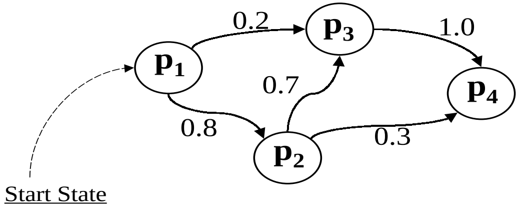
- States as nodes with names
- Transitions from states i-th and j-th as arcs labelled by conditional probabilities $P(X_{t+1} = p_j | X_t = p_i)$
  Note that 0 probability arcs are omitted from the graph.

| | $S_1$ | $S_2$ |
|---|---|---|
| $S_1$ | 0.70 | 0.30 |
| $S_2$ | 0.50 | 0.50 |

# Representation of a Markov Chain

## Graphical Representation

$$P(X_1 = p_1) = 1 \qquad \leftarrow StartState$$
$$P(X_k = p_3 | X_{k-1} = p_2) = 0.7 \qquad \forall k$$
$$P(X_k = p_4 | X_{k-1} = p_1) = 0 \qquad \forall k$$



# A Simple Example of Hidden Markov Model

Crazy Coffee Machine

- Two states: Tea Preferring (*TP*), Coffee Preferring (*CP*)
- Switch from one state to another randomly
- Simple (or visible) Markov model:
  **Iff** the machine output *Tea* in *TP* AND *Coffee* in *CP*

What we need is a description of the random event of switching from one state to another. More formally we need for each time step $n$ and couple of states $p_i$ and $p_j$ to determine following conditional probabilities:

$$P(X_{n+1} = p_j | X_n = p_i)$$

where $p_t$ is one of the two states *TP*, *CP*.

# *A Simple Example of Hidden Markov Model*

## *Crazy Coffee Machine*

Assume, for example, the following state transition model:

| | $TP$ | $CP$ |
|---|---|---|
| $TP$ | 0.70 | 0.30 |
| $CP$ | 0.50 | 0.50 |

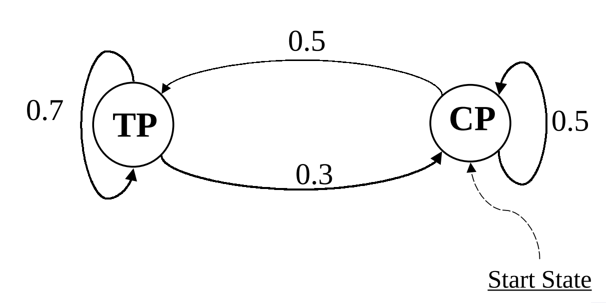
and let $CP$ be the starting state (i.e. $\pi_{CP} = 1$, $\pi_{TP} = 0$).

Potential Use:

1. What is the probability at time step 3 to be in state $TP$?
2. What is the probability at time step $n$ to be in state $TP$?
3. What is the probability of the following sequence in output: $(Coffee, Tea, Coffee)$?

# Crazy Coffee Machine

## Graphical Representation



# Crazy Coffee Machine

## Solution to Problem 1:

$P(X_3 = TP) =$ (given by $(CP, CP, TP)$ *and* $(CP, TP, TP)$)
$= P(X_1 = CP) \cdot P(X_2 = CP|X_1 = CP) \cdot P(X_3 = TP|X_1 = CP, X_2 = CP) +$
$+ \quad P(X_1 = CP) \cdot P(X_2 = TP|X_1 = CP) \cdot P(X_3 = TP|X_1 = CP, X_2 = TP) =$
$= P(CP)P(CP|CP)P(TP|CP, CP) + P(CP)P(TP|CP)P(TP|CP, TP) =$
$= P(CP)P(CP|CP)P(TP|CP) + P(CP)P(TP|CP)P(TP|TP) =$
$= 1 \cdot 0.50 \cdot 0.50 + 1 \cdot 0.50 \cdot 0.70 = 0.25 + 0.35 = 0.60$

# Crazy Coffee Machine

## Solution to Problem 2

$$P(X_n = TP) =$$
$$\sum_{CP,p_2,p_3,\ldots,TP} P(X_1 = CP)P(X_2 = p_2|X_1 = CP)P(X_3 = p_3|X_1 = CP, X_2 = p_2) \cdot \ldots \cdot P(X_n = TP|X_1 = CP, X_2 = p_2, \ldots, X_{n-1} = p_{n-1}) =$$
$$= \sum_{CP,p_2,p_3,\ldots,TP} P(CP)P(p_2|CP)P(p_3|p_2) \cdot \ldots \cdot P(TP|p_{n-1}) =$$
$$= \sum_{CP,p_2,p_3,\ldots,TP} P(CP) \cdot \prod_{t=1}^{n-2} P(p_{t+1}|p_t) \cdot P(p_n = TP|p_{n-1})$$
$$(= \sum_{p_1,\ldots,p_n} P(p_1) \cdot \prod_{t=1}^{n-1} P(p_{t+1}|p_t))$$

# Crazy Coffee Machine

*Solution to Problem 3:*

$$P(Cof, Tea, Cof) =$$
$$= P(Cof) \cdot P(Tea|Cof) \cdot P(Cof|Tea) = 1 \cdot 0.5 \cdot 0.3 = 0.15$$

# *A Simple Example of Hidden Markov Model (2)*

Crazy Coffee Machine

- **Hidden** Markov model: If the machine output *Tea*, *Coffee* or *Capuccino* **independently** from *CP* and *TP*.

What we need is a description of the random event of output(ting) a drink.

# *Crazy Coffee Machine*

A description of the random event of output(ting) a drink. Formally we need (for each time step $n$ and for each kind of output $O = \{Tea, Cof, Cap\}$), the following conditional probabilities:

$$P(O_n = o_k | X_n = p_i, X_{n+1} = p_j)$$

where $o_k \in \{Tea, Coffee, Capuccino\}$. This matrix is called the **output matrix** of the machine (or of its Hidden markov Model).

# A Simple Example of Hidden Markov Model (2)

Crazy Coffee Machine
Given the following output probability for the machine

| | Tea | Coffee | Capuccino |
|---|---|---|---|
| TP | 0.8 | 0.2 | 0.0 |
| CP | 0.15 | 0.65 | 0.2 |

and let *CP* be the starting state (i.e. $\pi_{CP} = 1, \pi_{TP} = 0$).

- Find the following probabilities of output from the machine

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  1. $(Cappuccino, Coffee)$ given that the state sequence is $(CP, TP, TP)$

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- Find the following probabilities of output from the machine
  1. $(Cappuccino, Coffee)$ given that the state sequence is $(CP, TP, TP)$
  2. $(Tea, Coffee)$ for any state sequence

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Crazy Coffee Machine
Given the following output probability for the machine

| | Tea | Coffee | Capuccino |
|---|---|---|---|
| TP | 0.8 | 0.2 | 0.0 |
| CP | 0.15 | 0.65 | 0.2 |

and let *CP* be the starting state (i.e. $\pi_{CP} = 1$, $\pi_{TP} = 0$).

- Find the following probabilities of output from the machine
  1. $(Cappuccino, Coffee)$ given that the state sequence is $(CP, TP, TP)$
  2. $(Tea, Coffee)$ for any state sequence
  3. a generic output $O = (o_1, \ldots, o_n)$ for *any* state sequence

# A Simple Example of Hidden Markov Model (2)

Solution for the problem 1 For the given state sequence $X = (CP, TP, TP)$
$P(O_1 = Cap, O_2 = Cof, X_1 = CP, X_2 = TP, X_3 = TP) =$
$P(O_1 = Cap, O_2 = Cof | X_1 = CP, X_2 = TP, X_3 = TP) P(X_1 = CP, X_2 = TP, X_3 = TP)) =$
$P(Cap, Cof | CP, TP, TP) P(CP, TP, TP))$

# A Simple Example of Hidden Markov Model (2)

Solution for the problem 1 For the given state sequence $X = (CP, TP, TP)$
$P(O_1 = Cap, O_2 = Cof, X_1 = CP, X_2 = TP, X_3 = TP) =$
$P(O_1 = Cap, O_2 = Cof | X_1 = CP, X_2 = TP, X_3 = TP)P(X_1 = CP, X_2 = TP, X_3 = TP)) =$
$P(Cap, Cof | CP, TP, TP)P(CP, TP, TP))$ Now:
$P(Cap, Cof | CP, TP, TP)$ is the probability of output $Cap, Cof$ *during* transitions from $CP$ to $TP$ and $TP$ to $TP$
and

# A Simple Example of Hidden Markov Model (2)

Solution for the problem 1 For the given state sequence
$X = (CP, TP, TP)$
$P(O_1 = Cap, O_2 = Cof, X_1 = CP, X_2 = TP, X_3 = TP) =$
$P(O_1 = Cap, O_2 = Cof | X_1 = CP, X_2 = TP, X_3 = TP) P(X_1 = CP, X_2 = TP, X_3 = TP)) =$
$P(Cap, Cof | CP, TP, TP) P(CP, TP, TP))$ Now:
$P(Cap, Cof | CP, TP, TP)$ is the probability of output $Cap, Cof$ *during* transitions from $CP$ to $TP$ and $TP$ to $TP$
and $P(CP, TP, TP)$ is the probability of the transition chain.
Therefore,

# A Simple Example of Hidden Markov Model (2)

Solution for the problem 1 For the given state sequence
$X = (CP, TP, TP)$
$P(O_1 = Cap, O_2 = Cof, X_1 = CP, X_2 = TP, X_3 = TP) =$
$P(O_1 = Cap, O_2 = Cof | X_1 = CP, X_2 = TP, X_3 = TP) P(X_1 = CP, X_2 = TP, X_3 = TP)) =$
$P(Cap, Cof | CP, TP, TP) P(CP, TP, TP))$ Now:
$P(Cap, Cof | CP, TP, TP)$ is the probability of output $Cap, Cof$ *during* transitions from $CP$ to $TP$ and $TP$ to $TP$
and $P(CP, TP, TP)$ is the probability of the transition chain.
Therefore,
$= P(Cap | CP, TP) P(Cof | TP, TP) =$(in our simplified model)
$= P(Cap | CP) P(Cof | TP) = 0.2 \cdot 0.2 = 0.04$

# A Simple Example of Hidden Markov Model (2)

Solutions for the problem 2

In general, for any sequence of three states $X = (X_1, X_2, X_3)$

$P(Tea, Cof | X_1, X_2, X_3) =$

$P(Tea, Cof) =$ (as sequences are a partition for the sample space)

$= \sum_{X_1, X_2, X_3} P(Tea, Cof | X_1, X_2, X_3) P(X_1, X_2, X_3)$ where

# A Simple Example of Hidden Markov Model (2)

Solutions for the problem 2

In general, for any sequence of three states $X = (X_1, X_2, X_3)$

$P(Tea, Cof|X_1, X_2, X_3) =$

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$P(Tea, Cof|X_1, X_2, X_3) = P(Tea|X_1, X_2) P(Cof|X_2, X_3) =$

(for the simplified model of the coffee machine )

$= P(Tea|X_1) P(Cof|X_2)$

# A Simple Example of Hidden Markov Model (2)

Solutions for the problem 2
In general, for any sequence of three states $X = (X_1, X_2, X_3)$
$P(Tea, Cof | X_1, X_2, X_3) =$
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$P(Tea, Cof | X_1, X_2, X_3) = P(Tea | X_1, X_2) P(Cof | X_2, X_3) =$
(for the simplified model of the coffee machine )
$= P(Tea | X_1) P(Cof | X_2)$ and (for the Markov constraint)
$P(X_1, X_2, X_3) = P(X_1) P(X_2 | X_1) P(X_3 | X_2)$

# A Simple Example of Hidden Markov Model (2)

Solutions for the problem 2

In general, for any sequence of three states $X = (X_1, X_2, X_3)$

$P(Tea, Cof|X_1, X_2, X_3) =$

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$= \sum_{X_1, X_2, X_3} P(Tea, Cof|X_1, X_2, X_3) P(X_1, X_2, X_3)$ where

$P(Tea, Cof|X_1, X_2, X_3) = P(Tea|X_1, X_2) P(Cof|X_2, X_3) =$

(for the simplified model of the coffee machine )

$= P(Tea|X_1) P(Cof|X_2)$ and (for the Markov constraint)

$P(X_1, X_2, X_3) = P(X_1) P(X_2|X_1) P(X_3|X_2)$

The simplified model is concerned with only the following transition chains

$(CP, CP, CP)$, $(CP, TP, CP)$, $(CP, CP, TP)$

$(CP, TP, TP)$

# A Simple Example of Hidden Markov Model (2)

Solutions for the problem 2

In general, for any sequence of three states $X = (X_1, X_2, X_3)$

The following probability is given

$P(Tea, Cof) =$

$$P(Tea|CP)P(Cof|CP)P(CP)P(CP|CP)P(CP|CP) + \quad \text{st.: } (CP, CP, CP))$$
$$P(Tea|CP)P(Cof|TP)P(CP)P(TP|CP)P(CP|TP) + \quad \text{st.: } (CP, TP, CP))$$
$$P(Tea|CP)P(Cof|CP)P(CP)P(CP|CP)P(TP|CP) + \quad \text{st.: } (CP, CP, TP))$$
$$P(Tea|CP)P(Cof|TP)P(CP)P(TP|CP)P(TP|TP) = \quad \text{st.: } (CP, TP, TP))$$

# A Simple Example of Hidden Markov Model (2)

Solutions for the problem 2

In general, for any sequence of three states $X = (X_1, X_2, X_3)$

The following probability is given

$$
\begin{aligned}
P(Tea, Cof) = \\
& P(Tea|CP)P(Cof|CP)P(CP)P(CP|CP)P(CP|CP)+ && \text{st.: } (CP,CP,CP)) \\
& P(Tea|CP)P(Cof|TP)P(CP)P(TP|CP)P(CP|TP)+ && \text{st.: } (CP,TP,CP)) \\
& P(Tea|CP)P(Cof|CP)P(CP)P(CP|CP)P(TP|CP)+ && \text{st.: } (CP,CP,TP)) \\
& P(Tea|CP)P(Cof|TP)P(CP)P(TP|CP)P(TP|TP) = && \text{st.: } (CP,TP,TP))
\end{aligned}
$$

$$
\begin{aligned}
&= 0.15 \cdot 0.65 \cdot 1 \cdot 0.5 \cdot 0.5+ \\
&+ \quad 0.15 \cdot 0.2 \cdot 1 \cdot 0.5 \cdot 0.3+ \\
&+ \quad 0.15 \cdot 0.65 \cdot 1 \cdot 0.5 \cdot 0.5+ \\
&+ \quad 0.15 \cdot 0.2 \cdot 1.0 \cdot 0.5 \cdot 0.7 =
\end{aligned}
$$

# A Simple Example of Hidden Markov Model (2)

Solutions for the problem 2

In general, for any sequence of three states $X = (X_1, X_2, X_3)$

The following probability is given

$$P(Tea, Cof) =$$

$$\begin{array}{ll} P(Tea|CP)P(Cof|CP)P(CP)P(CP|CP)P(CP|CP)+ & \text{st.: } (CP,CP,CP)) \\ P(Tea|CP)P(Cof|TP)P(CP)P(TP|CP)P(CP|TP)+ & \text{st.: } (CP,TP,CP)) \\ P(Tea|CP)P(Cof|CP)P(CP)P(CP|CP)P(TP|CP)+ & \text{st.: } (CP,CP,TP)) \\ P(Tea|CP)P(Cof|TP)P(CP)P(TP|CP)P(TP|TP) = & \text{st.: } (CP,TP,TP)) \end{array}$$

$$\begin{aligned} &= 0.15 \cdot 0.65 \cdot 1 \cdot 0.5 \cdot 0.5+ \\ &+ \quad 0.15 \cdot 0.2 \cdot 1 \cdot 0.5 \cdot 0.3+ \\ &+ \quad 0.15 \cdot 0.65 \cdot 1 \cdot 0.5 \cdot 0.5+ \\ &+ \quad 0.15 \cdot 0.2 \cdot 1.0 \cdot 0.5 \cdot 0.7 = \end{aligned}$$

$$\begin{aligned} &= 0.024375 + 0.0045 + 0.024375 + 0.0105 = \\ &= 0.06375 \end{aligned}$$

# A Simple Example of Hidden Markov Model (2)

Solution to the problem 3 (*Likelihood*)
In the general case, a sequence of $n$ symbols $O = (o_1, \ldots, o_n)$ out from any sequence of $n+1$ transitions $X = (p_1, \ldots, p_{n+1})$ can be predicted by the following probability:

$$
\begin{aligned}
P(O) &= \sum_{p_1,\ldots,p_{n+1}} P(O|X)P(X) = \\
&= \sum_{p_1,\ldots,p_{n+1}} P(CP) \prod_{t=1}^{n} P(O_t|p_t, p_{t+1}) P(p_{t+1}|p_t)
\end{aligned}
$$

# Modeling linguistic tasks as Stochastic Processes

## Advantages

There are several advantages to model a linguistic problem as an HMM

- It is a powerful mathematical framework for modeling
- It provides clear problems settings for different applications: **estimation**, **decoding** and **model induction**
- HMM-based models provides sound solutions for the above applications

We will see an example as the HMM modeling of POS tagging

# *Fundamental problems for HMM*

## *Fundamental Questions for HMM*

The complexity of training and decoding can be limited by the use of optimization techniques

- Given the observation sequence $O = O_1, ..., O_n$ and a model $\lambda = (E, T, \pi)$, how to efficiently compute $P(O|\lambda)$? (*Language Modeling*)
- Given the observation sequence $O = O_1, ..., O_n$ and a model $\lambda = (E, T, \pi)$, how do we choose the optimal state sequence $Q = q_1, ..., q_n$ responsible of generating O ? (*Tagging/Decoding*)
- How to adjust model parameters $\lambda = (E, T, \pi)$ so to maximize $P(O|\lambda)$? (*Model Induction*)

# *HMM: Mathematical Methods*

All the above problems can be approached by several optimization techniques able to limit the complexity.

- Language Modeling via *dynamic programming* (**Forward algorithms**) ($O(n)$)
- Tagging/Decoding via *dynamic programming* ($O(n)$) (**Viterbi**)
- Parameter estimation via *entropy minimization* (the *EM* algorithm)

A relevant issue is the availability of source data: supervised training cannot be applied always

# *The task of POS tagging*

### *POS tagging*

Given a sequence of morphemes $w_1, \ldots, w_n$ with ambiguous syntactic descriptions (i.e.part-of-speech tags) $t_j$, compute the sequence of $n$ POS tags $t_{j_1}, \ldots, t_{j_n}$ that characterize correspondingly all the words $w_i$.

# The task of POS tagging

## POS tagging

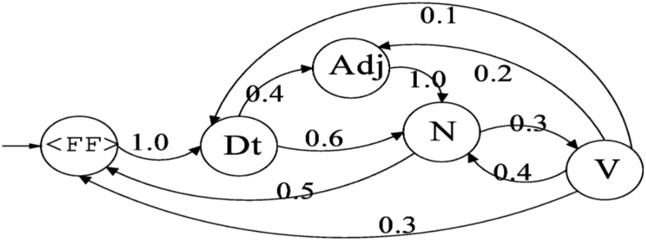
Given a sequence of morphemes $w_1, \ldots, w_n$ with ambiguous syntactic descriptions (i.e.part-of-speech tags) $t_j$, compute the sequence of $n$ POS tags $t_{j_1}, \ldots, t_{j_n}$ that characterize correspondingly all the words $w_i$.

Examples:

- *Secretariat is expected to race tomorrow*
- $\Rightarrow$ `NNP VBZ VBN TO VB NR`
- $\Rightarrow$ `NNP VBZ VBN TO NN NR`

# The task of POS tagging

## An example




| Emission probabilities | . | the | this | cat | kid | eats | runs | fish | fresh | little | big |
|---|---|---|---|---|---|---|---|---|---|---|---|
| <FF> | 1.0 | | | | | | | | | | |
| Dt | | 0.6 | 0.4 | | | | | | | | |
| N | | | | 0.6 | 0.1 | | | 0.3 | | | |
| V | | | | | | 0.7 | 0.3 | | | | |
| Adj | | | | | | | | | 0.3 | 0.3 | 0.4 |

# HMM and POS tagging

Given a sequence of morphemes $w_1, \ldots, w_n$ with ambiguous syntactic descriptions (i.e.part-of-speech tags), derive the sequence of $n$ POS tags $t_1, \ldots, t_n$ that maximizes the following probability:

$$P(w_1, \ldots, w_n, t_1, \ldots, t_n)$$

that is

$$(t_1, \ldots, t_n) = argmax_{pos_1, \ldots, pos_n} P(w_1, \ldots, w_n, pos_1, \ldots, pos_n)$$

# *HMM and POS tagging*

Given a sequence of morphemes $w_1, \ldots, w_n$ with ambiguous syntactic descriptions (i.e.part-of-speech tags), derive the sequence of $n$ POS tags $t_1, \ldots, t_n$ that maximizes the following probability:

$$P(w_1, \ldots, w_n, t_1, \ldots, t_n)$$

that is

$$(t_1, \ldots, t_n) = argmax_{pos_1, \ldots, pos_n} P(w_1, \ldots, w_n, pos_1, \ldots, pos_n)$$

Note that this is equivalent to the following:
$(t_1, \ldots, t_n) = argmax_{pos_1, \ldots, pos_n} P(pos_1, \ldots, pos_n | w_1, \ldots, w_n)$
as: $\frac{P(w_1, \ldots, w_n, pos_1, \ldots, pos_n)}{P(w_1, \ldots, w_n)} = P(pos_1, \ldots, pos_n | w_1, \ldots, w_n)$
and $P(w_1, \ldots, w_n)$ is the same for all the sequencies $(pos_1, \ldots, pos_n)$.

# HMM and POS tagging

## How to map a POS tagging problem into a HMM

The above problem

$$(t_1, \ldots, t_n) = argmax_{pos_1,\ldots,pos_n} P(pos_1, \ldots, pos_n | w_1, \ldots, w_n)$$

can be also written (Bayes law) as:

$$(t_1, \ldots, t_n) = argmax_{pos_1,\ldots,pos_n} P(w_1, \ldots, w_n | pos_1, \ldots, pos_n) P(pos_1, \ldots, pos_n)$$

# HMM and POS tagging

The HMM Model of POS tagging:

- **HMM States are mapped into POS tags** ($t_i$), so that $P(t_1, \ldots, t_n) = P(t_1)P(t_2|t_1)\ldots P(t_n|t_{n-1})$
- **HMM Output symbols are words**, so that $P(w_1, \ldots, w_n|t_1, \ldots, t_n) \quad = \quad \prod_{i=1}^{n} P(w_i|t_i)$
- Transitions represent moves from one word to another

Note that *the Markov assumption is used*

- to model probability of a tag in position $i$ (i.e. $t_i$) only by means of the preceeding part-of-speech (i.e. $t_{i-1}$)
- to model probabilities of words (i.e. $w_i$) based only on the tag ($t_i$) appearing in that position ($i$).

# HMM and POS tagging

The final equation is thus:

$$
\begin{aligned}
(t_1, \ldots, t_n) = \quad & argmax_{t_1, \ldots, t_n} P(t_1, \ldots, t_n | w_1, \ldots, w_n) = \\
& argmax_{t_1, \ldots, t_n} \prod_{i=1}^{n} P(w_i | t_i) P(t_i | t_{i-1})
\end{aligned}
$$

# Fundamental Questions for HMM in POS tagging

1. Given a model what is the probability of an output sequence, $O$:
   *Computing Likelihood, Language Modeling*.
2. Given a model and an observable output sequence $O$ (i.e. words), how to determine the sequence of states $T = (t_1, \ldots, t_n)$ such that it is the best explanation of the observation $O$:
   *Decoding Problem*
3. Given a sample of the output sequences and a space of possible models how to find out the best model, that is the model that best explains the data:
   *how to estimate parameters?*

# Fundamental Questions for HMM in POS tagging

- 1. (Language Modeling) Not much relevant for POS tagging, where $(w_1, \ldots, w_n)$ are always known.
  Trellis and dynamic programming technique.
- 2. (Decoding) Viterbi Algorithm for evaluating $P(T|O)$.
  Linear in the sequence length.
- 3. Baum-Welch (or Forward-Backward algorithm), that is a special case of Expectation Maximization estimation.
  Weakly supervised or even unsupervised.
  *Problems*: Local minima can be reached when initial data are poor.

# HMM and POS tagging

Advantages for adopting HMM in POS tagging

- An elegant and sound theory
- Training algorithms:
    - Estimation via EM (Baum-Welch)
    - Unsupervised (or possibly weakly supervised)
- Fast Inference algorithms: Viterbi algorithm
  Linear wrt the sequence length ($O(n)$)
- Sound methods for comparing different models and estimations
  (e.g. cross-entropy)

# Forward algorithm

In computing the likelihood $P(O)$ of an observation we need to sum up the probability of all paths in a Markov model. Brute force computation is not applicable in most cases.

The forward algorithm is an application of dynamic programming.

# Forward algorithm

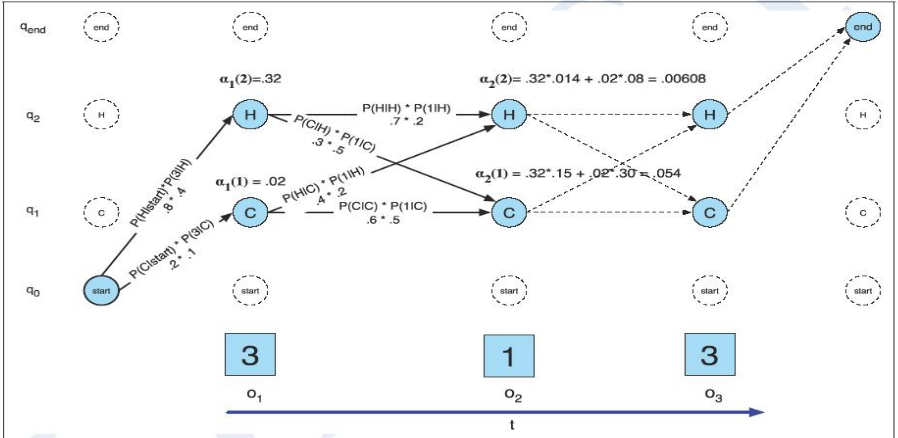



**Figure 6.6** The forward trellis for computing the total observation likelihood for the ice-cream events *3 1 3*. Hidden states are in circles, observations in squares. White (unfilled) circles indicate illegal transitions. The figure shows the computation of $\alpha_t(j)$ for two states at two time steps. The computation in each cell follows Eq˙ 6.11: $\alpha_t(j) = \sum_{i=1}^{N-1} \alpha_{t-1}(i) a_{ij} b_j(o_t)$. The resulting probability expressed in each cell is Eq˙ 6.10: $\alpha_t(j) = P(o_1, o_2 \dots o_t, q_t = j|\lambda)$.

# HMM and POS tagging: Forward Algorithm

**function** FORWARD(*observations* of len *T*, *state-graph*) **returns** *forward-probability*

*num-states* ← NUM-OF-STATES(*state-graph*)
Create a probability matrix *forward[num-states+2,T+2]*
*forward[0,0]* ← 1.0
**for** each time step *t* **from** 1 **to** *T* **do**
  **for** each state *s* **from** 1 **to** *num-states* **do**
    $forward[s,t] \leftarrow \sum_{1 \le s' \le num\text{-}states} forward[s',t-1] * a_{s',s} * b_s(o_t)$
**return** the sum of the probabilities in the final column of *forward*

**Figure 6.8** The forward algorithm; we've used the notation *forward*$[s,t]$ to represent $\alpha_t(s)$.

1. Initialization:

$$\alpha_1(j) \;=\; a_{0j}b_j(o_1) \;\; 1 \le j \le N \tag{6.12}$$

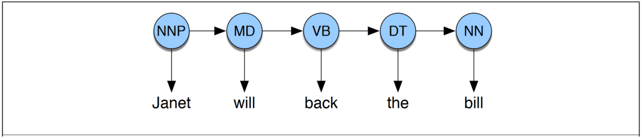
2. Recursion (since states 0 and N are non-emitting):

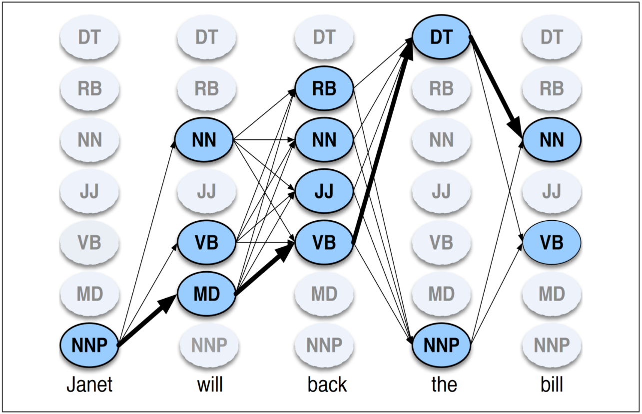
$$\alpha_t(j) = \sum_{i=1}^{N-1} \alpha_{t-1}(i)a_{ij}b_j(o_t); \quad 1 < j < N, 1 < t < T \tag{6.13}$$

3. Termination:

$$P(O|\lambda) = \alpha_T(N) = \sum_{i=2}^{N-1} \alpha_T(i)\, a_{iN} \tag{6.14}$$

# *Decoding: the Viterbi algorithm*



# *Viterbi algorithm*

In decoding we need to find the most likely state sequence given an observation $O$. The Viterbi algorithm follows the same approach (dynamic programming) of the Forward.

Viterbi scores are attached to each possible state in the sequence.

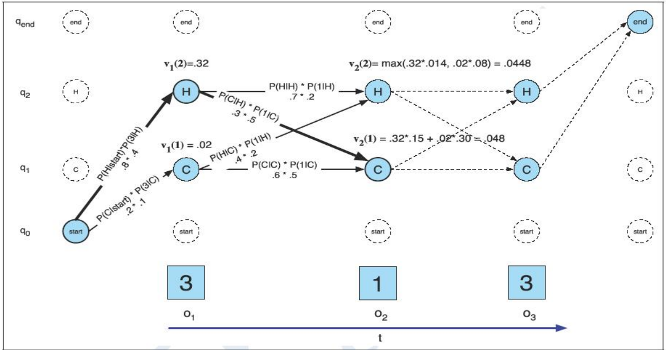



**Figure 6.9** The Viterbi trellis for computing the best path through the hidden state space for the ice-cream eating events *3 1 3*. Hidden states are in circles, observations in squares. White (unfilled) circles indicate illegal transitions. The figure shows the computation of $v_t(j)$ for two states at two time steps. The computation in each cell follows Eq˙ 6.10: $v_t(j) = \max_{1 \leq i \leq N-1} v_{t-1}(i)\, a_{ij}\, b_j(o_t)$ The resulting probability expressed in each cell is Eq˙ 6.16: $v_t(j) = P(q_0, q_1, \ldots, q_{t-1}, o_1, o_2, \ldots, o_t, q_t = j|\lambda)$.

# HMM and POS tagging: the Viterbi Algorithm

**function** VITERBI(*observations* of len *T*,*state-graph*) **returns** *best-path*

*num-states* ← NUM-OF-STATES(*state-graph*)
Create a path probability matrix *viterbi[num-states+2,T+2]*
*viterbi[0,0]* ← 1.0
**for** each time step *t* **from** 1 **to** *T* **do**
  **for** each state *s* **from** 1 **to** *num-states* **do**
    $viterbi[s,t] \leftarrow \max_{1 \le s' \le num\text{-}states} viterbi[s',t-1] * a_{s',s} * b_s(o_t)$
    $backpointer[s,t] \leftarrow \operatorname{argmax}_{1 \le s' \le num\text{-}states} viterbi[s',t-1] * a_{s',s}$
Backtrace from highest probability state in final column of *viterbi[]* and return path

**Figure 6.10** Viterbi algorithm for finding optimal sequence of tags. Given an observation sequence and an HMM $\lambda = (A, B)$, the algorithm returns the state-path through the HMM which assigns maximum likelihood to the observation sequence. Note that states 0 and N+1 are non-emitting *start* and *end* states.

# HMM and POS tagging: Parameter Estimation

Supervised methods in tagged data sets:

- Output probs: $P(w_i|p^j) = \frac{C(w_i, p^j)}{C(p^j)}$
- Transition probs: $P(p^i|p^j) = \frac{C(p^i \quad \text{follows} \quad \text{p}^\text{j})}{C(p^j)}$
- Smoothing: $P(w_i|p^j) = \frac{C(w_i, p^j)+1}{C(p^j)+K^i}$
  (see Manning& Schutze, Chapter 6)

# HMM and POS tagging: Parameter Estimation

Unsupervised (few tagged data available):

- With a dictionary: $P(w_i|p^j)$ are early estimated from $D$, while $P(p^i|p^j)$ are randomly assigned
- With equivalence classes $u_L$, (Kupiec92):

  $$P(w^i|p^L) = \frac{\frac{1}{|L|}C(u^L)}{\sum_{u_{L'}} \frac{C(u^{L'})}{|L'|}}$$

  For example, if $L$ ={noun, verb} then $u_L = \{cross, drive, \ldots\}$

# HMM and POS tagging: Equivalence classes (Kupiec '92)

*J. Kupiec*

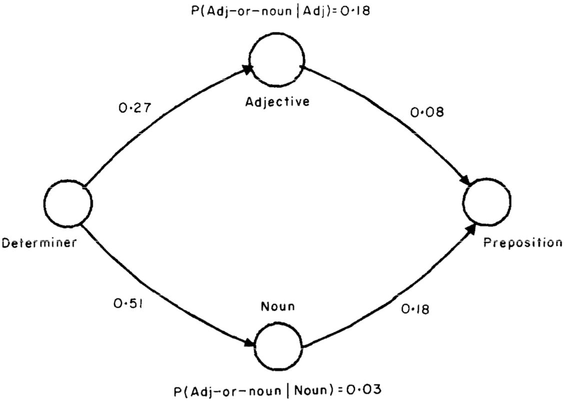



**Figure 4.** Probabilities for adjective/noun paths.

# POS tagging: References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# Exercise

Consider a two-bit register. The register has four possible states: `00`, `01`, `10` and `11`. Initially, at time `0`, the contents of the register is chosen at random to be one of these four states, each with equal probability. At each time step, beginning at time `1`, the register is randomly manipulated as follows: with probability `1/2`, the register is left unchanged; with probability `1/4`, the two bits of the register are exchanged (e.g., `01` becomes `10`); and with probability `1/4`, the right bit is flipped (e.g., `01` becomes `00`). After the register has been manipulated in this fashion, the left bit is observed. Suppose that on the first three time steps, we observe `0, 0, 1`.

- Formulate the register as an HMM. What is the probability of transitioning from every state to every other state? What is the probability of observing output (0 or 1) in each state?
- What is the probability of being in each state at time $t$ after observing only the first $t$ bits, for $t = 1, 2, 3$.